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# Algorithms for Graph Partitioning

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## **Abstract**

Graph partitioning is a fundamental algorithmic problem in combinatorics motivated by applications such as clustering and community detection in social networks as well as theoretical importance in particular in spectral methods and the Unique Games Conjecture in computational complexity.

The main theme in this problem is the following: Given a graph consisting of a collection of loosely connected dense subgraphs, design an efficient algorithm to detect, either exactly or approximately, the underlying dense subgraphs. There are many variations of the problem, ranging from information theoretic possibility of discovering the communities to efficient algorithms for doing so as well as local walk and distributed models for the algorithmic task.

In this research, a systematic study of the major techniques and discoveries in this area has been conducted with an emphasis on the methods based on the spectral graph theory. In the last years, spectral graph partitioning approaches have become very popular and there has been a growing interest in their applications, mainly on account of their efficiency and mathematical elegance. Therefore, the main concepts of the spectral partitioning algorithms are comprehensively discussed in this research and some novel applications of these methods have been concluded at the end.

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